

PROPERTIES OF FRACTIONAL EXCLUSION STATISTICS IN INTERACTING PARTICLE SYSTEMS

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Abstract. We show that fractional exclusion statistics is manifested in general in interacting systems and we discuss the conjecture recently introduced (J. Phys. A: Math. Theor. **40**, F1013, 2007), according to which if in a thermodynamic system the mutual exclusion statistics parameters are not zero, then they have to be proportional to the dimension of the Hilbert space on which they act. By using simple, intuitive arguments, but also concrete calculations in interacting systems models, we show that this conjecture is not some abstract consequence of unphysical modelling, but is a natural—and for a long time overlooked—property of fractional exclusion statistics. We show also that the fractional exclusion statistics is the consequence of interaction between the particles of the system and it is due to the change from the description of the system in terms of free-particle energies, to the description in terms of the quasi-particle energies. From this result, the thermodynamic equivalence of systems of the same, constant density of states, but any exclusion statistics follows immediately.

Key words: Quantum statistics, ensemble equivalence, thermodynamic equivalence

1 INTRODUCTION

Fractional exclusion statistics (FES) is a generalization of Bose and Fermi statistics, introduced by Haldane in Ref. [1], and with its thermodynamic properties calculated mainly by Isakov [2] and Wu [3]. The concept have been applied to a large number of systems of interacting particles [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and describes quasiparticles that exist in finite dimensional Hilbert spaces—in general, quasiparticles in a finite region of condensed matter; the Hilbert spaces are extensive, increasing proportionally to the size of the condensed matter region [1]. Quasiparticles that exist in these Hilbert spaces are called in general *species*—each subspace contains one species. Let us denote the Hilbert spaces by \mathcal{H}_i , with $i = 0, 1, \dots$, each of them containing N_i particles and G_i available states [15]. Then, by increasing the number of particles of one species, say N_i increases to $N_i + \delta N_i$, the number of available states in any of the Hilbert spaces changes by $\delta G_j = -\alpha_{ji}\delta N_i$. The proportionality factors, α_{ij} , are called *direct* (when $i = j$) and *mutual* (when $i \neq j$) *exclusion statistics parameters*.

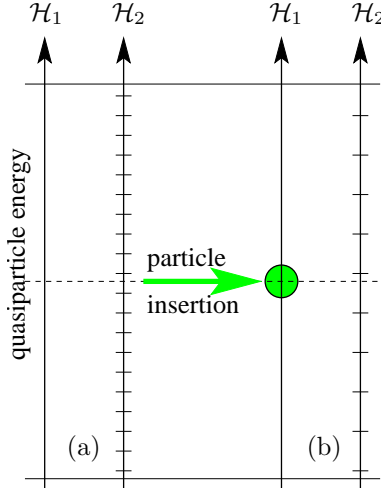


Figure 1: In (a), \mathcal{H}_1 and \mathcal{H}_2 are two Hilbert spaces defined by the ranges of quasiparticle energies (the energy levels in \mathcal{H}_1 are not relevant, and therefore are not represented). In (b), extra particles are inserted into \mathcal{H}_1 , which changes the density of states in \mathcal{H}_2 and therefore the number of allowed single-particle states. This is a manifestation of fractional exclusion statistics.

The simplest examples are the Bose and Fermi statistics. For ideal bosons, $\alpha_{ij} = 0$ for any i and j , whereas for ideal fermions $\alpha_{ii} = 1$ for any i and $\alpha_{ij} = 0$ for any i and j , if $i \neq j$.

2 PROPERTIES OF THE MUTUAL EXCLUSION STATISTICS PARAMETERS

The FES (other than Bose and Fermi statistics) is the result of interaction between quasiparticles. The finite dimensional Hilbert spaces involved, \mathcal{H}_i , are formed of quasiparticle wavefunctions corresponding to eigenvalues contained in finite regions of the phase space. For example Isakov [2] and Iguchi and Sutherland [16, 17] define the Hilbert spaces by the wavevector eigenvalues, \mathbf{k} , whereas Murthy and Shankar [10], Sen and Bhaduri [11], Hansson, Leinaas, and Viefers [12] define the Hilbert spaces by the quasiparticle energy eigenvalues, $\tilde{\epsilon}$. In all the cases, the eigenvalues of the wavefunctions contained in any of the Hilbert spaces belong to a finite range (or a multi-dimensional volume); therefore the quasiparticle-quasiparticle interaction, which changes the density of states (DOS) in the phase-space, changes also the number of quasiparticle states in the range, leading to FES [18] (see figure 1).

Let's take for example a system where the quasiparticle energies may be written as [10, 11, 12, 7, 19, 20, 18]

$$\tilde{\epsilon}_i = \epsilon_i + \sum_{j=0}^{i-1} V_{ij} n_j + \frac{1}{2} V_{ii} n_i, \quad (1)$$

where ϵ_i are the energies of the noninteracting particles; The total energy of the system is $E = \sum_{i=0}^{\infty} n_i \tilde{\epsilon}_i$. We assume that the system is large enough—and the energy levels dense enough—to introduce the (quasi)continuous density of states, $\sigma(\epsilon)$ and write equation (1) as

$$\tilde{\epsilon} = \epsilon + \int_0^{\epsilon} V(\epsilon, \epsilon') n(\epsilon') \sigma(\epsilon') d\epsilon', \quad (2)$$

where we replaced the subscripts i and j of V_{ij} by the corresponding energy levels, ϵ and ϵ' , respectively. In the variable $\tilde{\epsilon}$, the density of states is different from $\sigma(\epsilon)$ and we shall denote it by $\tilde{\sigma}(\tilde{\epsilon})$. To calculate $\tilde{\sigma}(\tilde{\epsilon})$, we take a small interval, $\delta\epsilon$, containing $\sigma(\epsilon) \cdot \delta\epsilon$ states, and transform it into the interval $\delta\tilde{\epsilon}$, which, obviously, will contain the same number of states. Dividing the number of states by the energy interval and using equation (2), we find

$$\tilde{\sigma}[\tilde{\epsilon}(\epsilon)] = \frac{\sigma(\epsilon)}{1 + \int_0^{\epsilon} \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon' + V(\epsilon, \epsilon' \nearrow \epsilon) \sigma(\epsilon) n(\epsilon)} \quad (3)$$

In figure 2 we show an example. In some arbitrary, dimensionless units, with the population of the single particle energy levels shown in the left plot, the DOS $\sigma(\epsilon)$ shown as the solid line in the middle plot, and constant interaction potential, $V(\epsilon, \epsilon') \equiv V$, we calculate $\tilde{\sigma}[\tilde{\epsilon}(\epsilon)]$ (point line in the middle plot) and $\tilde{\epsilon}$ (point line in the right plot)—for the concrete calculations of figure 2 we choose $V = 1$, $\sigma(\epsilon) = \sqrt{\epsilon}$, and $n(\epsilon) = [\exp(\epsilon + 1) - 1]^{-1}$.

To show how FES is manifested in the system, we split the quasiparticle energy axis into small intervals, $[\tilde{\epsilon}_0, \tilde{\epsilon}_1], \dots, [\tilde{\epsilon}_{i-1}, \tilde{\epsilon}_i], \dots$ (as shown in figure 3), so that each interval contains large enough numbers of particles and available single particle states; we denote by $N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}) \equiv N_i$ the number of particles in the interval $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$,

$$N_i \equiv N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}) = \int_{\tilde{\epsilon}_i}^{\tilde{\epsilon}_{i+1}} \tilde{\sigma}(\tilde{\epsilon}) n(\tilde{\epsilon}) d\tilde{\epsilon} = \int_{\epsilon_i}^{\epsilon_{i+1}} \sigma(\epsilon') n(\epsilon') d\epsilon',$$

and, assuming that the energy intervals are small enough so that we can replace in the integral of equation (2) $V(\epsilon_M, \epsilon)$ by $V(\epsilon_M, \epsilon_{i-1})$ for any $\epsilon \in (\epsilon_i, \epsilon_{i+1}]$, we write

$$\tilde{\epsilon}_j = \epsilon_j + \sum_{i=0}^{j-1} V(\epsilon_j, \epsilon_i) N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}). \quad (4)$$

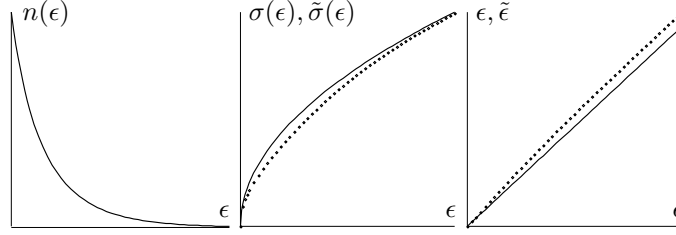


Figure 2: The DOS on the quasiparticle energy axis, $\tilde{\sigma}[\tilde{\epsilon}]$ —the point line in the middle plot—, and the quasiparticle energy, $\tilde{\epsilon}$ —the point line in the right plot—for a bosonic single particle energy levels population, $n(\epsilon) = [\exp(\epsilon + 1) - 1]^{-1}$, a DOS $\sigma(\epsilon) = \sqrt{\epsilon}$ and a constant interaction potential, $V(\epsilon, \epsilon') = 1$. In the right plot we show both, $\tilde{\epsilon}(\epsilon)$ (point line) and $\epsilon(\epsilon)$ (solid line) for comparison.

If we insert δN_i particles into the interval $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$ and we keep all the free particle energy levels $\epsilon_{j(>i)}$ unchanged, then we change the quasiparticle energy levels $\tilde{\epsilon}_{j(>i)}$ by $\delta \tilde{\epsilon}_{j(>i)} = V(\epsilon_j, \epsilon_i) \delta N_i$ (see the small, round, arrows on the $\tilde{\epsilon}$ axis in figure 3). In this way, in all the intervals $[\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}]$, with $j \neq i$, the N_j and G_j remain unchanged. Therefore, if we maximize the partition function with respect to the population of the energy intervals $[\epsilon_i, \epsilon_{i+1}]$, $i = 0, 1, \dots$, we have to take into account the change of the quasiparticle energies due to the change of the populations. We shall come back to this method later.

Another method to calculate the partition function and its maximum, is to fix the intervals $[\tilde{\epsilon}_0, \tilde{\epsilon}_1], \dots, [\tilde{\epsilon}_{j-1}, \tilde{\epsilon}_j], \dots$ along the $\tilde{\epsilon}$ axis. In this case, the insertion of the δN_i particles into the interval $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$ changes the values of ϵ_j (for $j > i$) given by equation (4) into ϵ'_j , which should be calculated like in [18]:

$$\tilde{\epsilon}_j = \epsilon'_j + V(\epsilon'_j, \epsilon_i) I_i + \sum_{k=0}^i V[\epsilon'_j, \epsilon_k] N(\tilde{\epsilon}_k, \tilde{\epsilon}_{k+1}) + \sum_{k=i+1}^j V[\epsilon'_j, \epsilon'_k] N(\tilde{\epsilon}_k, \tilde{\epsilon}_{k+1}). \quad (5)$$

Because $\sigma(\epsilon)$ is independent of the population, the change of $\epsilon_{j(>i)}$ (see the $\delta \epsilon_j$ s on the left axis in figure 3) leads to a change in the number of available single particle states in the interval $[\epsilon'_j, \epsilon'_{j+1}]$ and therefore a change of the number of states in the interval $[\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}]$ and a change of $\tilde{\sigma}(\tilde{\epsilon})$, according to equation (3). *This is the manifestation of FES.*

The exclusion statistics parameters of the FES gas have been calculated in Ref. [18]. There we showed in the general case that these parameters obey the properties conjectured in Ref. [15], namely that the mutual exclusion statistics parameters are proportional to the dimension of the Hilbert space on which they act. Let us take as example a system of *bosons* of constant

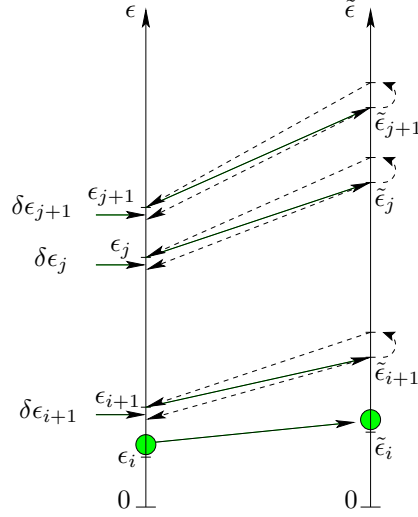


Figure 3: The intervals $[\epsilon_0, \epsilon_1], \dots, [\epsilon_{j-1}, \epsilon_j], \dots$, on the ϵ axis, and the corresponding intervals, $[\tilde{\epsilon}_0, \tilde{\epsilon}_1], \dots, [\tilde{\epsilon}_{j-1}, \tilde{\epsilon}_j], \dots$ on the $\tilde{\epsilon}$ axis. The insertion of particles in the interval $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$ changes the intervals on the ϵ axis, from $[\epsilon_i, \epsilon_{i+1}]$, above, if the intervals on the $\tilde{\epsilon}$ axis are held fixed.

interaction potential, $V(\epsilon, \epsilon') = V$. Then, the direct and mutual exclusion statistics parameters are [18]

$$\tilde{\alpha}_{\tilde{\epsilon}\tilde{\epsilon}} = \alpha_{\tilde{\epsilon}\tilde{\epsilon}} = V\sigma[\epsilon(\tilde{\epsilon})], \quad (6a)$$

and

$$\tilde{\alpha}_{\tilde{\epsilon}\epsilon'} = \theta(\tilde{\epsilon} - \epsilon')V \left. \frac{d\{\ln[\sigma(\epsilon)]\}}{d\epsilon} \right|_{\epsilon(\tilde{\epsilon})} \cdot \sigma[\epsilon(\tilde{\epsilon})]\delta\epsilon \equiv \alpha_{\tilde{\epsilon}\epsilon'}\delta G(\epsilon), \quad (6b)$$

respectively, where $\delta\epsilon$ is the energy interval in which the mutual statistics is manifested and $\delta G = \sigma[\epsilon(\tilde{\epsilon})]\delta\epsilon$ is the number of energy levels contained in it. Having the exclusion statistics parameters (6), we can write the integral equation (19) of Ref. [15] for the most probable particle population as:

$$\beta[\mu - \tilde{\epsilon}(\epsilon)] + \ln \frac{[1 + n(\epsilon)]^{1-V\sigma(\epsilon)}}{n(\epsilon)} = V \int_{\epsilon}^{\infty} \ln[1 + n(\epsilon')] \left[\frac{d\sigma}{d\epsilon'} \right] d\epsilon'. \quad (7)$$

Differentiating both sides of (7) with respect to ϵ , we obtain the differential equation,

$$\frac{dn}{d\epsilon} \cdot \frac{1 + Vn(\epsilon)\sigma(\epsilon)}{n(\epsilon)[1 + n(\epsilon)]} = -\beta \frac{d\tilde{\epsilon}}{d\epsilon}. \quad (8)$$

If we use (2) into (8), the differential equation reduces to

$$\beta^{-1} \frac{dn(\epsilon)}{d\epsilon} = -n(\epsilon)[1 + n(\epsilon)], \quad (9)$$

which, provided that $n(\epsilon)$ is positive and converges to zero at $\epsilon \rightarrow \infty$, admits only the solution

$$n(\epsilon) = \{\exp[\beta(\epsilon - \mu')] - 1\}^{-1}. \quad (10a)$$

This is, of course, the Bose population in the variable ϵ , and relation (10a) is true for any $\sigma(\epsilon)$ and constant V . To determine μ' , we plug (10a) back into (7) and obtain

$$\mu' = \mu - VN, \quad (10b)$$

where $N = \int_0^\infty \sigma(\epsilon)n(\epsilon)d\epsilon$ is the total number of particles in the system. From $n(\epsilon)$, the population $n(\tilde{\epsilon})$ follows by a simple change of variable, given by equation (1).

Equations (10) may seem surprising, but let's have now another perspective on the problem. Since V is a constant, the total energy of the system reduces to

$$E = \sum_i \epsilon_i n_i + \frac{VN^2}{2}, \quad (11)$$

which is the energy of the gas in the mean-field approximation. This case is easy to treat also from the perspective of the free particle energies, ϵ . If we turn back to the division of the energy axis ϵ into the intervals $[\epsilon_i, \epsilon_{i+1})$, $i = 0, 1, \dots$, of $N_i = \int_{\epsilon_i}^{\epsilon_{i+1}} n(\epsilon)\sigma(\epsilon)d\epsilon$ particles and $G_i = \int_{\epsilon_i}^{\epsilon_{i+1}} \sigma(\epsilon)d\epsilon$ states, we write the partition function and we maximize it with respect to the populations $n(\epsilon)$, then we get

$$n[\epsilon'(\epsilon)] = \{\exp[\beta(\epsilon' - \mu)] - 1\}^{-1}, \quad (12a)$$

which is exactly equation (10a), but with a redefinition of the quasiparticle energy,

$$\epsilon' = \epsilon + VN. \quad (12b)$$

The quasiparticle energies (12b) are different from the ones defined in (1). Moreover, $\epsilon'_i - \epsilon_i$ is independent of i , unlike $\tilde{\epsilon}_i - \epsilon_i$, which is $\sum_{j=0}^{i-1} V_{ij}n_j + \frac{1}{2}V_{ii}n_i$ and depends on i . Another difference is that while the total energy of the system can be written as $E = \sum_i \tilde{\epsilon}_i n_i$, the summation $\sum_i \epsilon'_i n_i$ gives $E + VN^2/2$.

So, the definition of quasiparticle energies (1) leads to the manifestation of FES along the quasiparticle energy axis, $\tilde{\epsilon}$. If V is constant, the FES particle distribution, $n(\tilde{\epsilon})$, corresponds to the Bose distribution in the free-particle energies, $n(\epsilon)$ (equations 10). If $\sigma(\epsilon)$ is also constant, then the mutual exclusion statistics parameters, $\alpha_{\tilde{\epsilon}\tilde{\epsilon}'}$ (equation 6b) vanishes and the direct exclusion statistics parameters become independent of $\tilde{\epsilon}$: $\alpha_{\tilde{\epsilon}\tilde{\epsilon}} = V\sigma$; if $V\sigma = 1$, then the interacting Bose gas may be interpreted as a Fermi gas. From here, the thermodynamic equivalence of systems of the same, constant density of states and any exclusion statistics [21, 22, 19] follows directly.

3 CONCLUSIONS

This paper is a continuation of Ref. [18], where we showed that the fractional exclusion statistics (FES) is manifested in general in systems of interacting particles and the mutual exclusion statistics parameters are proportional to the dimension of the Hilbert space on which they manifest. Here we calculated the density of quasiparticle states in general and we analyzed in particular a system in which the inter-particle interaction potential does not depend on the particles quantum numbers ($V_{ij} \equiv V$). Using the formalism presented in Refs. [15, 18], we showed that in a gas of interacting bosons, the resulting FES quasiparticle population, obtained by the maximization of the partition function, is actually the original Bose distribution written as a function of the quasiparticle energy, $\tilde{\epsilon}$, instead of the free-particle energy, ϵ . Therefore the FES reduces to a change of variable, from ϵ —the free-particle energy—to $\tilde{\epsilon}$ —the quasiparticle energy—in the population of the single particle energy levels, n . From this result, the thermodynamic equivalence of systems of the same, constant, density of states but any exclusion statistics [21, 22, 19] follows immediately.

Acknowledgments

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4 REFERENCES

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